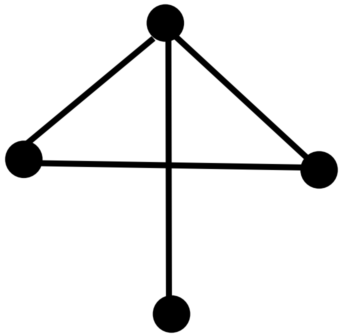
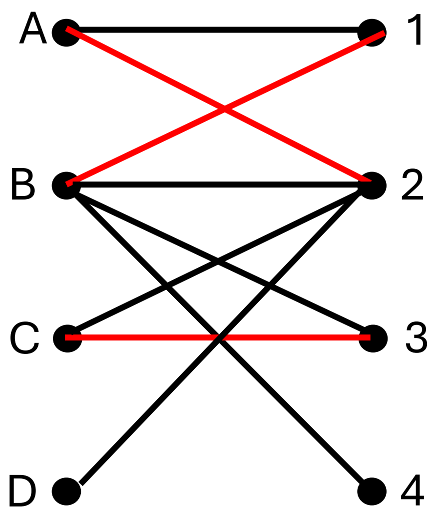
THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

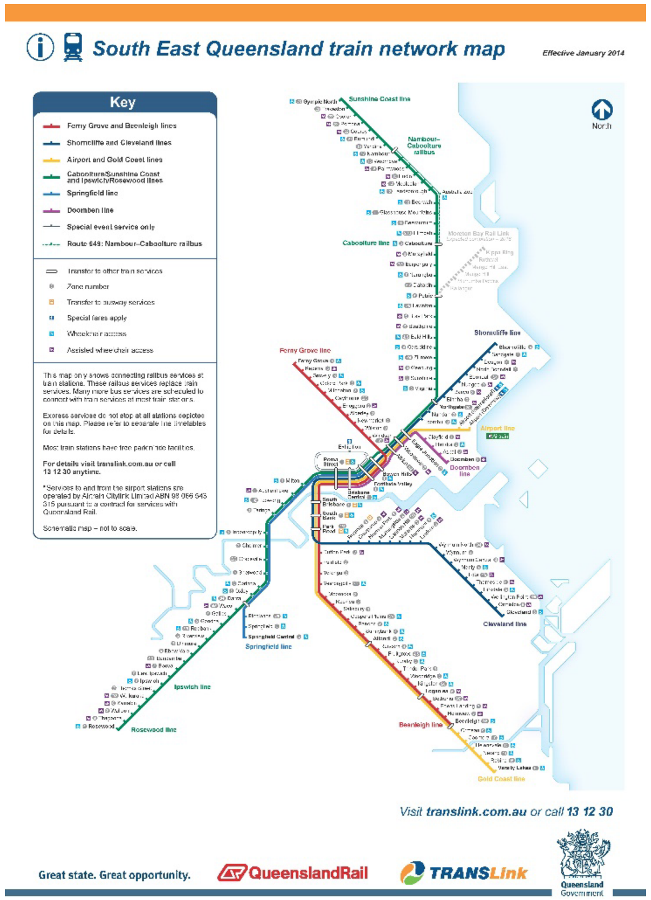
School of
Mathematics and Physics



Graphs and Networks

Presented by: Barbara Maenhaut, UQ





General Mathematics: Unit 4 Investing and networking

Topic 3: Graphs and Networks

- Graphs, associated terminology and the adjacency matrix (4 hours)
- Planar graphs, paths and cycles (8 hours)

Topic 4: Networks and decision mathematics 1

- Trees and minimum connector problems (4 hours)
- Project planning and scheduling using critical path analysis (8 hours)

Topic 5: Networks and decision mathematics 2

- Flow networks (4 hours)
- Assigning order and the Hungarian algorithm (7 hours)

Goal for today – to appreciate the versatility of networks as a modelling tool, and share some activity ideas that can be used to familiarise students with graph terminology and concepts.

Graphs and networks – models for everything and anything

Planar graphs

- Designing printed circuits
- Infrastructure where crossings are expensive including train tracks and road networks
- Data visualisation

Spanning trees

- Design of efficient networks for telecommunication, electrical, etc.
- Computer science – networking protocols, routing algorithms
- Cluster analysis in data

Critical path analysis

- Project scheduling and resource allocation
- Task prioritisation
- Risk management

Max flow min cut

- Transportation and logistics
- Maximising data throughput in computer networks
- Image segmentation algorithms to partition into foreground and background

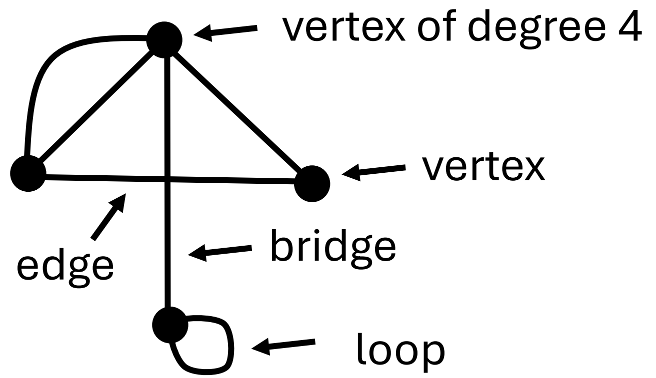
Hungarian algorithm

- Resource allocation, scheduling, transportation and logistics
- Multi-object tracking in computer vision
- Maximise training performance in machine learning

Other examples

- Journey planning in public transportation
- Use of Eulerian trails and Hamiltonian paths in DNA sequencing
- Modelling disease spread and using cuts to stop the spread.

There is a lot of new terminology for students!!



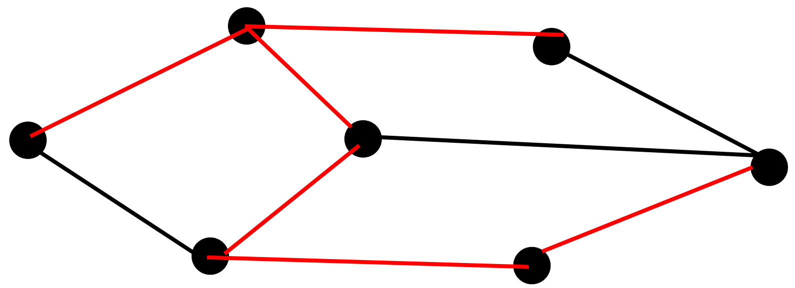
graph, vertex (node), edge (arc), loop, degree of a vertex, subgraph, bridge

simple graph, complete graph, bipartite graph, directed graph (digraph), weighted graph, connected graph, planar graph

walk, trail, path, open walk, open trail, open path, closed walk, closed trail (circuit), closed path (cycle)

Eulerian trail, semi-Eulerian graph, Eulerian circuit, Eulerian graph

Hamiltonian path, semi-Hamiltonian graph, Hamiltonian cycle, Hamiltonian graph



tree, spanning tree, minimum spanning tree

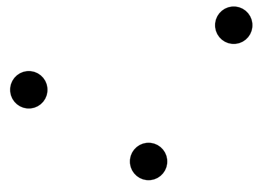
source node, sink node, cut, minimum cut and maximum flow

Activity 1 – Sprouts 2-player game using pen and paper

https://nrich.maths.org/games/sprouts

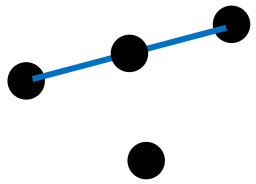
Invented in 1967 by two mathematicians, John H. Conway and Michael S. Paterson

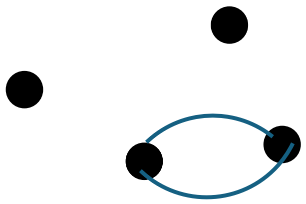
Start by drawing any number of vertices, for example 3.



A valid move – join two vertices with an edge and place a new vertex in the middle of the new edge OR join a vertex to Itself with a loop and place a new vertex in the middle of the loop, subject to the following restrictions:

- The resulting graph must be planar (no edge may cross another edge)
- The maximum degree of any vertex is 3 (you cannot create a vertex of degree 4 or more).



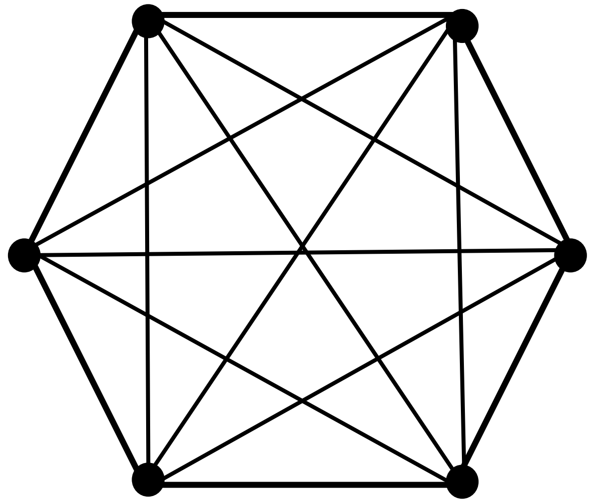


Players take turns adding edges until they get stuck, the winner is the last person to add an edge.

Game with n vertices ends after m moves, creating a graph with r faces.

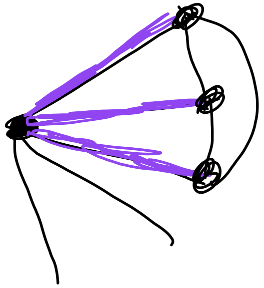
$2n \leq m \leq 3n-1$, $r \leq 2n+1$, see https://nrich.maths.org/articles/sprouts-explained

Activity 2a – Sim 2-player game using pen and paper

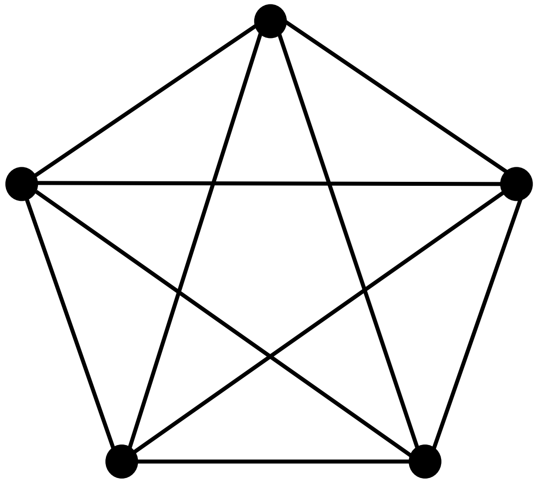


Start with the complete graph of order 6.
Two players take turns colouring any uncoloured edges. One player colours in red, and the other colours in blue, with each player trying to avoid the creation of a triangle made solely of their colour (only triangles with the vertices as corners count; intersections of edges are not relevant). A player who creates a triangle in their colour loses.

Cannot end in a draw.
- vertex of degree 5 has at least 3 edges of the same colour.
- consider the triangle on those 3 vertices



Activity 2b – a variant of Sim



Start with the complete graph of order 5.
Two players take turns colouring any uncoloured edges. One player colours in red, and the other colours in blue, with each player trying to create a triangle made solely of their colour (only triangles with the vertices as corners count; intersections of edges are not relevant); the player who first completes such a triangle wins. The game is declared a draw if all ten edges have been coloured without a player winning.

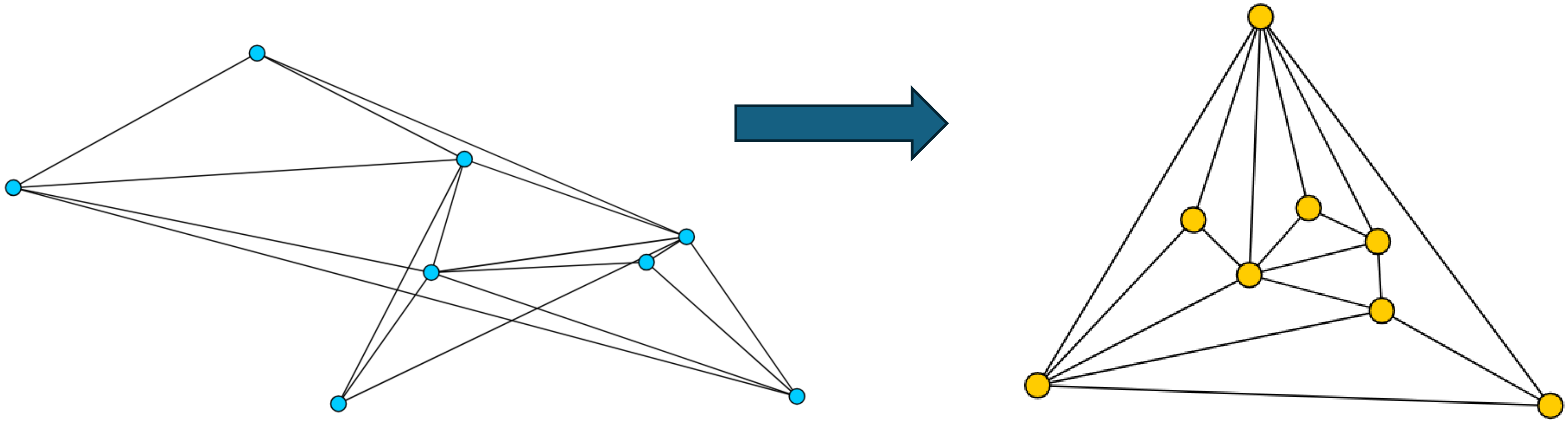
There is a winning strategy for the player who goes first.

Activity 3 – Planarity 1-player online game

Consider K_4

A 2005 puzzle computer game programmed by John Tantalo. The original website is no longer working, but you can play at https://www.jasondavies.com/planarity/

In the planarity game, the player is presented with a planar graph, drawn with many crossings.
The goal for the player is to eliminate all the crossings and construct a straight-line planar embedding of the graph by moving the vertices one by one into better positions.



Activity 4a – Shannon switching game 2-player game using pen and paper

Invented by the mathematician and electrical engineer Claude Shannone, prior to 1951
https://en.wikipedia.org/wiki/Shannon_switching_game

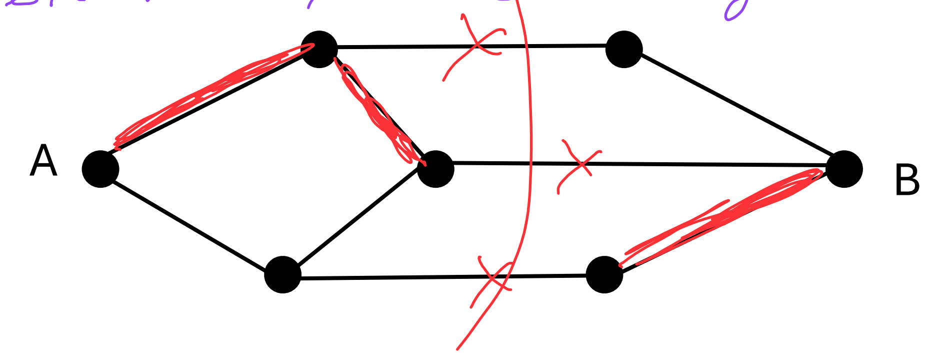
The game is played on a graph with two special vertices, A and B. One player has the goal of colouring edges of G to connect A to B by a path of edges of their colour. The other player aims to prevent this by deleting uncoloured edges of G to disconnect A from B. The players alternate turns, with one player colouring an edge that is still in the graph, and the other player deleting an uncoloured edge.

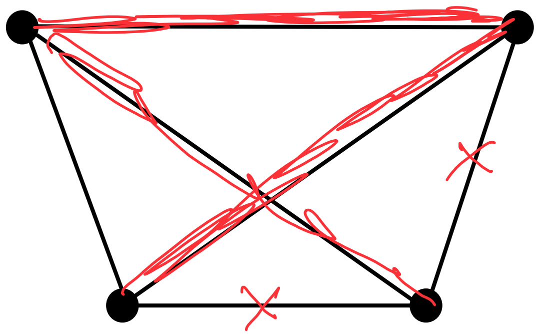
Activity 4b – Spanning tree game 2-player game using pen and paper

A variant of the Shannon switching game https://math.mit.edu/~goemans/18433S09/shannon-game.pdf

A variant in which the goal of the player colouring edges is to create a spanning tree of the given graph G, thereby connecting all pairs of vertices. The players alternate turns, with one player colouring any one edge that is still in the graph, and the other player deleting an uncoloured edge. The colouring player wins if they construct a spanning tree of the graph; otherwise, the deleting player wins.

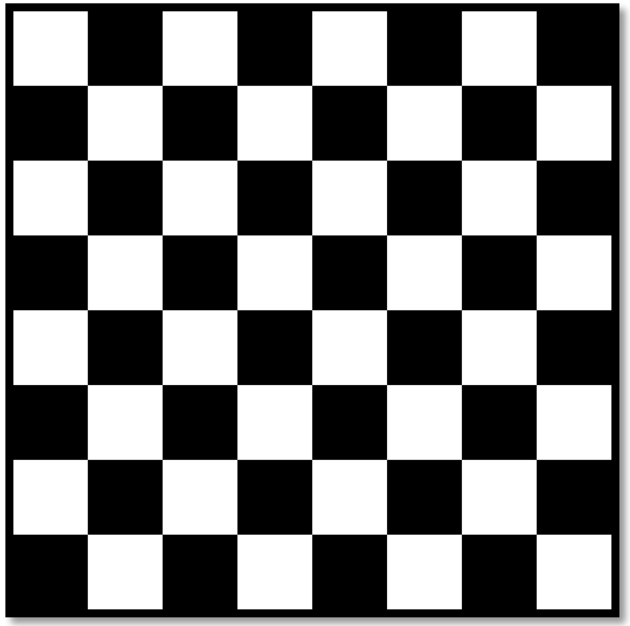
Deleter wins by deleting the edges of a cut.





Activity 5 – Knight's tour 1-player game

Can you find a sequence of moves of a knight on a chessboard such that the knight visits each square exactly once?



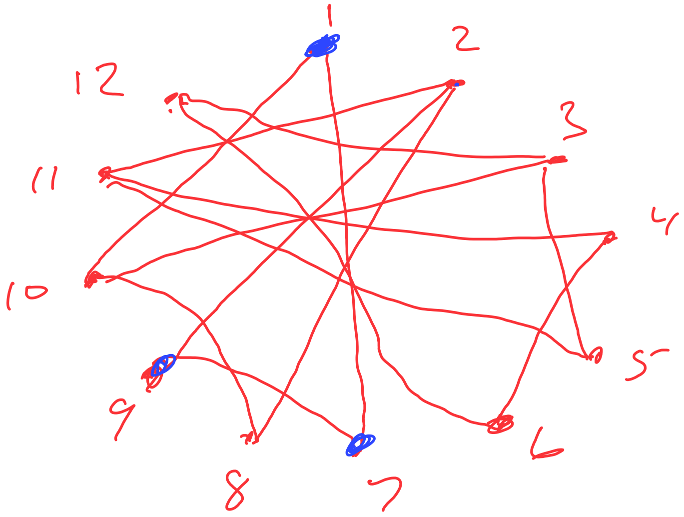
Knight's tour challenge: https://www.maths-resources.com/knights/

Animation:
https://upload.wikimedia.org/wikipedia/commons/d/da/Knight%27s_tour_anim_2.gif

Numberphile video demonstrating closed and open knight's tours:
https://www.youtube.com/watch?v=ab_dY3dZFHM

Try this problem on a 3 × 4 board.

1	2	3	4
5	6	7	8
9	10	11	12



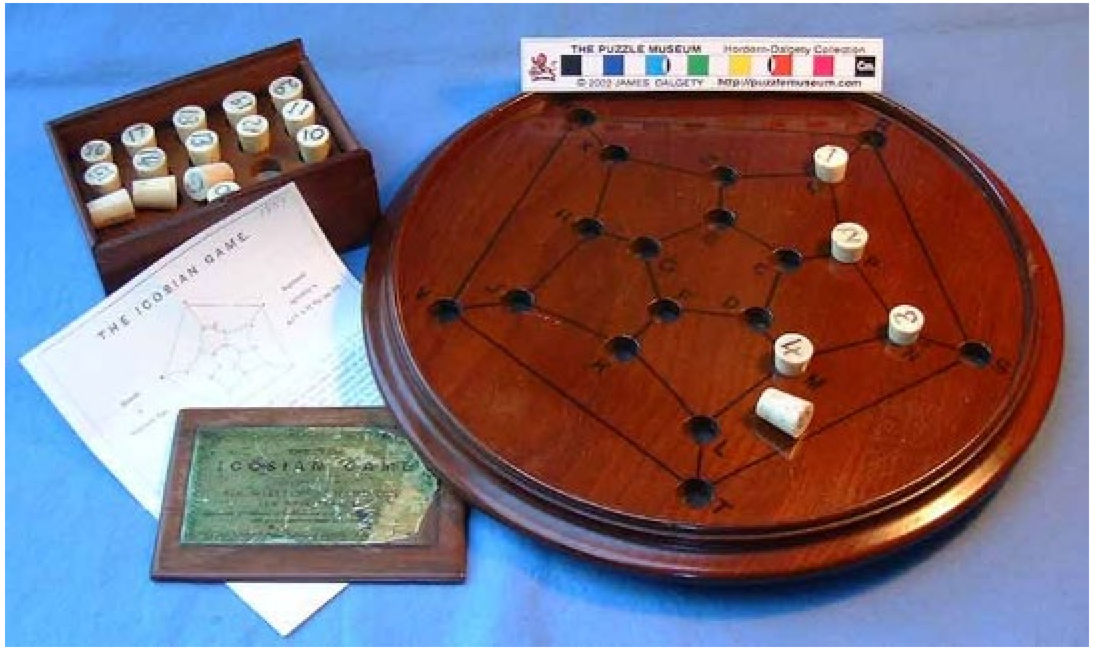
10, 1, 7, 9, 2,

Hamiltonian path

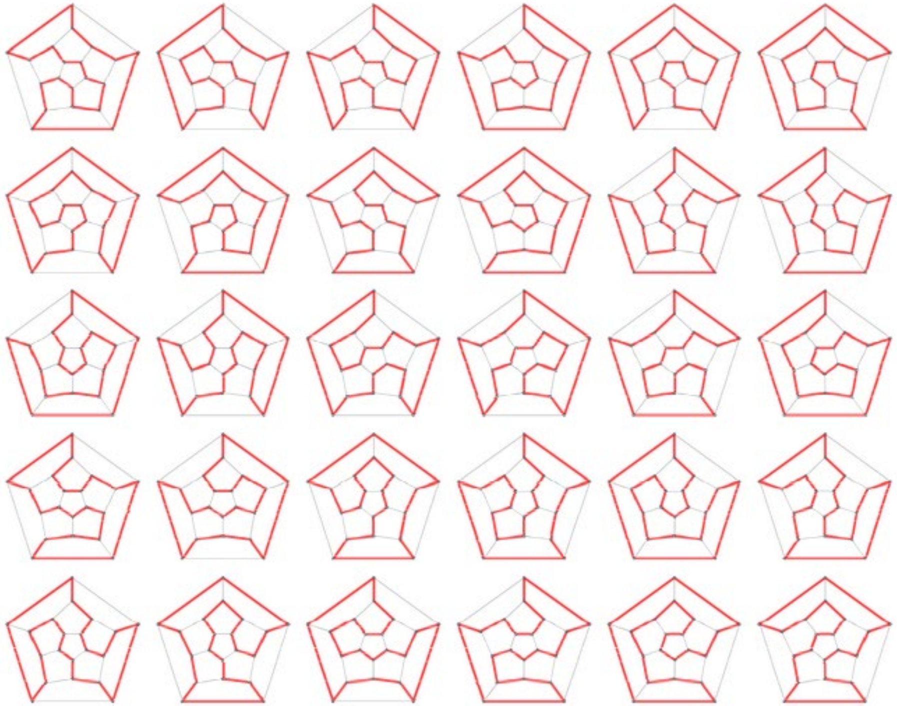
8, 10, 1, 7, 9, 2, 11, 5, 3, 12, 6, 4

A ***Hamilton cycle*** is a closed walk that visits every vertex exactly once. A graph that has a Hamilton cycle is ***Hamiltonian***.

Hamilton cycles are named after Sir William Rowan Hamilton. In 1857 he invented the Icosian game, in which players have to find a Hamilton cycle in the graph of the dodecahedron. He sold the game to a London games dealer in 1859 for 25 pounds, and the game was subsequently distributed in Europe in various forms, but was not very successful.



There are 30 solutions to the Icosian game.



You can play the icosian game at https://www.geogebra.org/m/u3xggkcj

The game on other graphs is available at:
https://ludii.games/details.php?keyword=Hamiltonian%20Maze

A **matching** in a graph is a set of edges in the graph, no two of which have a vertex in common.

Activity 6 – MaxMatch 2-player game using pen and paper

https://ludii.games/details.php?keyword=MaxMatch

The players take turns colouring edges of a given graph with the goal of increasing the size of their matching, or blocking their opponent from doing so. In each turn, the current player can choose to pass, or to colour an uncoloured edge. If the edge they select is disjoint from all the other edges they have previously coloured, then the player gets one point, if it is adjacent to an edge of their colour, they get no points. Play continues until all edges of the graph are coloured, or both players choose to pass. The player with the highest score wins.